The Relationship Between Temperature (°C) and Spring Constant of a Rubber Band ($\mathrm{Nm^{-1}}$)

IBIS Code: jgt165

Introduction

Materials crafted from natural and synthetic rubber are utilized in countless practical applications. The elastic trait allows for use in commercial products such as tubes, tires, washers, etc. In addition rubber's chemical composition embodies strengthened longevity by reducing mechanical stress in heavy industrial machinery namely electrical motors, engines, automobile parts, etc. When under use in the real world, such equipment is prone to strong doses of heat i.e. the burning of fuel in a car eventually dissipates as heat in where the elastic strength of rubber is critical for function. This paper will investigate the relationship between temperature and elasticity (specifically spring constant) through the research question:

Research Question

How does temperature T (°C) affect the spring constant k ($\mathrm{Nm^{-1}}$) of a rubber band?

Theoretical Background

The elastic property of a rubber/elastic band can be defined as the object's adaptability to renew to its original shape following distortion.

To start, the behaviour of elastic solids can be explained mathematically by applying Hooke's Law:

$$F_a = k\Delta x \tag{1}$$

where F_a is the applied force in newtons (N), k is the spring constant in newtons per meter ($\mathrm{Nm^{-1}}$) and Δx is the stretch or compression in meters (m).

To correlate elasticity with spring constant, we refer to the definition of the elastic property of a rubber/elastic band: the object's adaptability to renew to its original shape following distortion. Thus, a more elastic object will stretch a longer distance Δx and produce a smaller spring constant k as $k \propto \frac{1}{\Delta x}$ from Equation 1.

To calculate spring constant, assume two scenarios of the identical rubber band. One with natural length L_1, and another with a mass elongating it's length to L_2, as illustrated in the diagram below. The change in length $L_2 - L_1$ is defined to be ΔL.

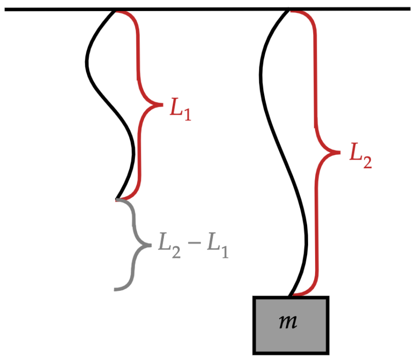


Figure 1: Mass Elongation

To calculate k, we allow for the applied force F_a to be equal to F_g where $F_g = mg$, (gravitational force by the mass m).

$$mg = k\Delta L \Rightarrow \boxed{k = \frac{mg}{\Delta L}}, \text{(where } g \text{ is } 9.81\text{ms}^{-1}) \tag{2}$$

To further explore and hypothesize the behaviour of elastic materials such as the rubber band, we introduce the concept of Young's Modulus, an extension of Hooke's Law which normalizes the dimensions of the sample.

Young's Modulus is defined as:

$$Y = \frac{\sigma}{\epsilon}$$

where σ is defined as the tensile stress of the sample; $\frac{F}{A}$ (force per unit area), and ϵ is the tensile strain; $\frac{\Delta L}{L}$ (relative change in length where L is original length), and Y representing the modulus of elasticity.

Substituting:

$$\begin{aligned} Y &= \frac{F/A}{\Delta L/L} = \frac{FL}{A\Delta L} \\ F &= \text{force applied in N} \\ A &= \text{cross sectional area in m}^2 \\ \Delta L &= \text{change in length in m} \\ L &= \text{original length in m} \end{aligned} \tag{3}$$

Using Young's Modulus, Hooke's Law can be rewritten as:

$$\text{Given } \sigma = \tfrac{F}{A} \text{ and } \epsilon = \tfrac{\Delta L}{L}, \text{ and } \sigma = Y\epsilon,$$

$$\tfrac{F}{A} = Y\tfrac{\Delta L}{L}, \text{ and with rearranging:}$$

$$F = \tfrac{YA}{L}\Delta L$$

This informs us that $k = \frac{AY}{L}$ (4). The notion of Young's Modulus can be investigated to provide a rough trend of Y with temperature.

Rubber bands are created of multiple Polymer molecules arranged in chain-like fashion in the underlying chemical structure. Most solid objects have a tendency to expand in length when prone to increased amounts of temperature. The aforementioned structure of Polymer has shown to produce a counter intuitive trend and shrink in the scenarios of high temperature. When heated, the entropy of the object is increased, and the chains get more tangled together, reducing in length.

Furthermore as the rubber band shrinks, according to R. L. Anthony, R. H Caston and Eugene Guth (1942), it is shown that the Young Modulus follows an inverse trend and rises in value. Suppose a scenario where the rubber band is kept at constant strain ϵ and the stress σ is measured with increasing temperature:

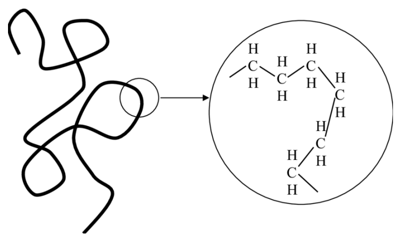


Figure 2: Polymer Chains (Jansen, 2007)

$$Y = \frac{\sigma_{\text{thermal}}}{\epsilon} \tag{5}$$

The force required to sustain the object has been shown to increase, thus increasing stress - σ_{thermal}. This shows that the tensile stress of the elastic object should be correlated to the temperature of the sample.

Let T be the temperature in °C. If a linear relationship between T and σ_{thermal} is to be assumed, the following summarizing proportionalities are formed:

$$k \propto Y \propto \sigma_{\text{thermal}} \propto T$$

$$\Rightarrow k \propto T \tag{6}$$

Hypothesis

From the summarizing proportionalities in Equation 6, we notice that an increase in temperature T results in greater stress that the rubber band experiences, ultimately increasing Y. As k and Y are directly proportional, k increases as well. With the mentioned observations, and consideration into assuming σ_{thermal} to be linearly correlated with T, the hypothesis is formulated as:

The spring constant k (Nm^{-1}) of the rubber band will increase linearly as a result of increasing temperature T (°C).

Variables

Independent Variable: The independent variable is the temperature T (°C) of the rubber band (°C ± 0.1 : $-5.0, 5.0, 15.0, 25.0, 35.0, 45.0, 55.0$) which was measured using a lab thermometer. The average room temperature is estimated as 25.0 °C, and there are 3 positive and negative increments on either side. Each increment was repeated for a total of 5 trials.

Dependent Variable: The dependent variable is the spring constant k (Nm^{-1}). As the force F_g applied onto the rubber band is constant, k is calculated by measuring the change in length ΔL and utilizing Equation 2. A ruler attached to the apparatus was used to measure length with uncertainty ($\pm 0.0005\,\mathrm{m}$). To elongate the rubber band, a hook weight of mass $m = 0.085\,\mathrm{kg}$ was used.

Controlled Variables:

Mass of Weight: The same hook weight with mass $m = 0.085\,\mathrm{kg}$ was used every iteration.

Rubber Band: Since each rubber band has its own unique structural composition, switching it for different increments would produce random error. Hence, the same initial rubber band was used throughout the experiment.

Environment: External factors such as wind blowing on the apparatus could affect the length the rubber band elongates by, thus becoming an extraneous variable. Hence, the experiment was conducted in a closed environment (limiting air passing in through windows).

Fixed Point: The rubber band must be secured to the apparatus, to avoid causing an external factor affecting elongation. Thus the rubber band was attached to a photo hook which was nailed into the retort stand.

Apparatus

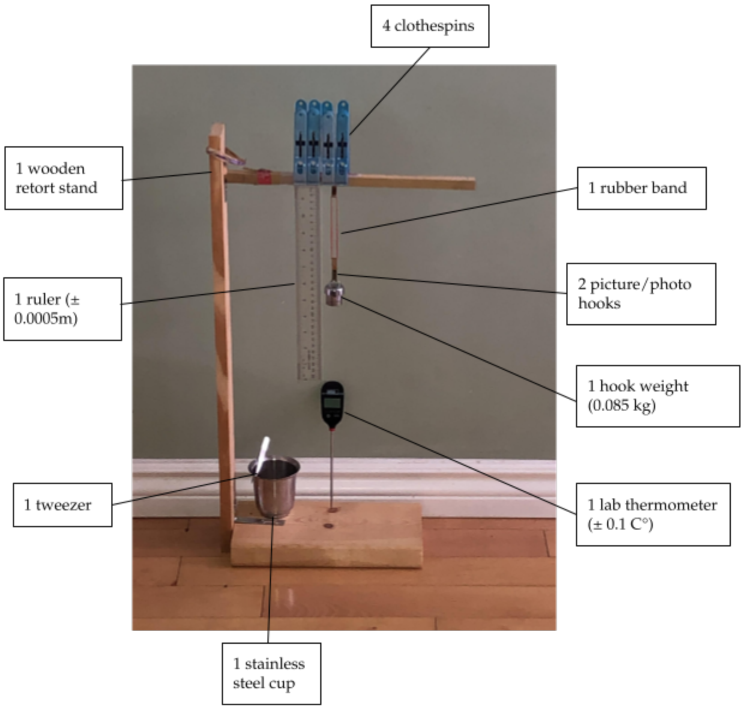


Figure 3: Full setup of retort stand and key materials

Additional materials not included in Figure 3 include a cooktop and a fridge, both intended for the thermal interaction of the rubber band.

Method

1. Set up the apparatus as shown in Figure 3.
2. Attach the hook weight to the rear end of the photo hook.
3. Place the rubber band onto the photo hook, and record initial length.
4. If heating the rubber band, add water to the stainless steel cup and insert the rubber band. Place onto a stove. If cooling, place the rubber band into a fridge, and assess temperature.
5. Maintain temperature using the lab thermometer and terminate when the desired value has been reached.
6. Attach the rubber band onto the photo hooks, and measure the change in length from the original value, with the ruler.
7. Repeat steps 4-6 for 5 trials of the current increment.

8. Repeat steps 4-7 for 7 increments of temperature ($-5.0, 5.0, 15.0, 25.0, 35.0, 45.0, 55.0$ in °C).

Safety, Ethical, and Environmental Concerns

Precautions such as wearing safety goggles, and oven mitts, must be obeyed during the procedure. When emptying hot water to repeat for another increment, caution is advised.

Data Collection, Processing, and Analysis

The determined value of $\overline{\Delta L}$ (distance of elongation), for a particular iteration, is calculated by averaging $\Delta L_i \forall i$, where i is the trial number $[1, 5]$. The uncertainty of $\overline{\Delta L}$ is calculated as $\frac{\max \Delta L_i - \min \Delta L_i}{2}$ (7), and denoted as $\overline{\Delta L}_{\text{unc}}$.

Sample Calculations

This sample will suppose the results with ($N = 5$) trials as follows and the uncertainty calculated using Equation 7. Assume initial length $L = (0.0210 \pm 0.0005)\,\text{m}$. When elongated, the change in length has it's uncertainty propagated to result in $\pm\,0.001\,\text{m}$. All ΔL_i are hence truncated to 3 decimal places.

$$\Delta L_1 = 0.023\,\text{m},\ \Delta L_2 = 0.022\,\text{m},\ \Delta L_3 = 0.024\,\text{m},\ \Delta L_4 = 0.023\,\text{m},\ \Delta L_5 = 0.023\,\text{m}$$

$$\overline{\Delta L} = \frac{1}{N} \cdot \sum_{i=1}^{N} \Delta L_i = 0.023\,\text{m}$$

$$\overline{\Delta L}_{\text{unc}} = \frac{\max \Delta L_i - \min \Delta L_i}{2} = 0.001\,\text{m}$$

To calculate the spring constant k, assume the mass of the hook weight to be $m = 0.085\,\text{kg}$. k can be calculated with Equation 2 and propagating uncertainties:

$$k = \frac{mg}{\overline{\Delta L}} = \frac{0.085\,\text{kg} \times 9.81\text{ms}^{-2}}{0.023\,\text{m}} \approx 36.3\,\text{Nm}^{-1}$$

$$\frac{\Delta k}{k} = \frac{\overline{\Delta L}_{\text{unc}}}{\overline{\Delta L}} \Rightarrow \Delta k = \frac{k \times \overline{\Delta L}_{\text{unc}}}{\overline{\Delta L}} = \frac{36.3\,\text{Nm}^{-1} \times 0.001\,\text{m}}{0.023\,\text{m}} = \pm\,1.578\,\text{Nm}^{-1} \approx \pm\,1.6\,\text{Nm}^{-1}$$

Raw Data

The initial length of the rubber band was measured with uncertainty $(0.095 \pm 0.0005)\,\mathrm{m}$. As the stretched length is measured, the change in length is calculated as ΔL, with uncertainty $2 \cdot 0.0005\,\mathrm{m} = \pm\,0.001\,\mathrm{m}$.

Temperature T ($\pm$ 0.1 °C)	Change in Length ΔL ($\pm$ 0.001m)				
	Trial 1	**Trial 2**	**Trial 3**	**Trial 4**	**Trial 5**
-5.0	0.028	0.028	0.028	0.029	0.029
5.0	0.025	0.024	0.025	0.026	0.025
15.0	0.023	0.022	0.023	0.023	0.024
25.0	0.021	0.021	0.021	0.020	0.020
35.0	0.020	0.019	0.018	0.019	0.018
45.0	0.018	0.016	0.017	0.017	0.016
55.0	0.016	0.014	0.015	0.015	0.015

Table 1: Raw quantitative data of the experiment

Processed Data

Temperature T ($\pm$ 0.1 °C)	Mean Change in Length $\overline{\Delta L}$ in m	Uncertainty in Mean Change in Length $\pm$ $\overline{\Delta L}_{unc}$ in m	Spring Constant k in Nm^{-1}	Uncertainty of Spring Constant $\pm$ Δk in Nm^{-1}
-5.0	0.0284	0.0005	29.36	0.52
5.0	0.025	0.001	33.4	1.3
15.0	0.023	0.001	36.3	1.6
25.0	0.0206	0.0005	40.48	0.98
35.0	0.019	0.001	44.4	2.4
45.0	0.017	0.001	49.6	3.0
55.0	0.015	0.001	55.6	3.7

Table 2: Processed quantitative data of the experiment

The absolute uncertainty $\overline{\Delta L}_{\text{unc}}$ were all rounded to one significant digit, and mean changes in length L truncated to the same number of decimal places as the corresponding uncertainty. The uncertainty of the spring constant was rounded to 2 significant figures, and corresponding spring constant k truncated to the same number of decimal places.

Qualitative Observations

When the rubber band was placed onto the retort stand, an evident slight oscillation was observed, but diminished over time. Furthermore, heating by convection (by the stove), took a significantly longer period of time in comparison to the cooling process (with the fridge).

Linear Model Graph

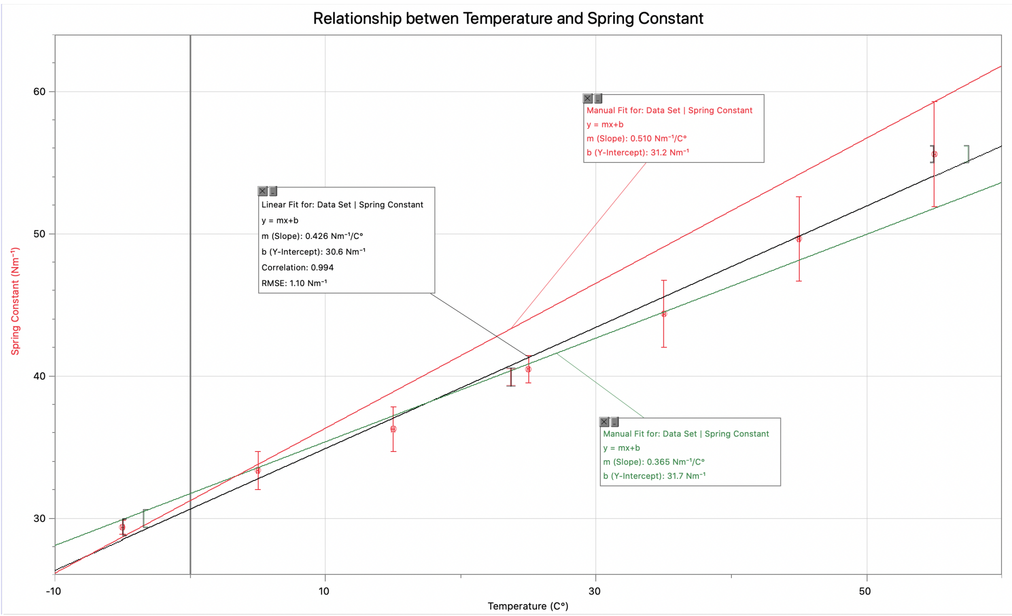


Figure 4: Linear model graph (Temperature °C vs Spring Constant Nm^{-1})

As processed data was entered into Vernier Logger Pro (see Appendix I), and a linear regression (best line fit) was performed, the above graph represents the results, with corresponding error bars and min-max lines. The correlation coefficient of the center linear fit is 0.994, hence indicating a strong linear fit between the independent and dependent variable. The trend displayed agrees with the hypothesis that the spring constant of the rubber band increases linearly with temperature. The vertical error bars intersect with the center line of best fit, and horizontal error bars are negligible as it was determined to be a constant value of $\pm 0.01°\text{C}$. Lastly, we can see that the slope of best-fit falls comfortably within the min and max slope lines; $0.365 \leq 0.426 \leq 0.510$.

Physical Interpretation

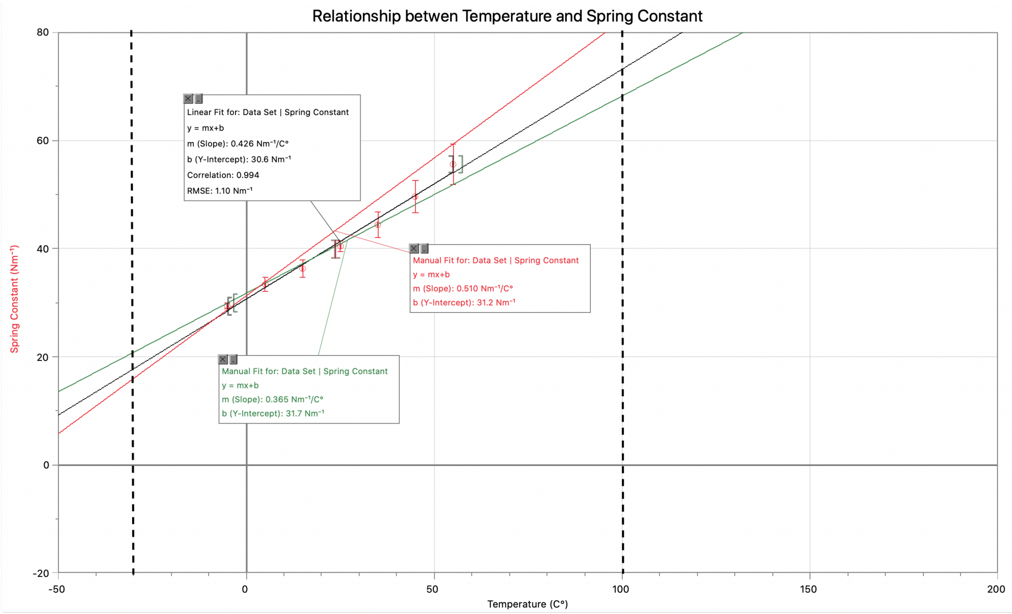


Figure 5: Further interpretation (Temperature °C vs Spring Constant Nm^{-1})

From the derived linear model in Figure 4, there are critical assumptions that deviate from the physical norm of a rubber band. Due to extreme temperatures, the rubber band may undergo physical deformations, representing the breaking point of a linear relationship. The above figure (Figure 5), represents the breaking point of Polymer placed under estimated temperatures of $-30\,°\mathrm{C}$ and $100\,°\mathrm{C}$, but ultimately representing a domain for T for some arbitrary range $[L, R]$ in °C. A final aspect to note is that as F_a and ΔL are the same parity from Equation 2, k cannot be negative. Hence it is important to note that k rather tends to $0\,\mathrm{Nm}^{-1}$ as $T \to -\infty$, rather than cross the x-axis (which is indicated by the linear model).

Conclusion

The investigation aims to examine the relationship between temperature (°C) and spring constant (Nm^{-1}) of a rubber band. It is hypothesized that the spring constant k (Nm^{-1}) of the rubber band will increase linearly as a result of increasing temperature T (°C) of the rubber band.

Through experimentation and data-processing, the line of best fit showed that a linear relationship is established between T and k, in that k increased linearly when measured against varying increments of T. The relationship is represented by the equation: $y = 0.426x + 30.6$ in Figure 5, verifying the hypothesis. The phenomenon can be explained by observing the chemical structure of Polymeric chains and relating it to the notion of elastic materials described by Young's Modulus and Hooke's Law.

Discussing the precision of the experiment, we analyze the line of best fit, as well as the magnitude of uncertainty present in the investigation. The line of best fit passes through 6/7 data points (within error bars), with exception to the first iteration. The precision of data is also prevalent in the coefficient of determination R^2 value, indicating the scatter of the data around the trend-line. In Figure 4, we can see the correlation value to be 0.994.

As the absolute uncertainty of $\overline{\Delta L}$ ($\overline{\Delta L}_{\text{unc}}$) in Table 2 is relatively constant for most increments, varying from either ± 0.005 or ± 0.001, the percentage uncertainty rises as T increases. From the equation to calculate propagated uncertainty we know:

$$\frac{\Delta k}{k} = \frac{\overline{\Delta L}_{\text{unc}}}{\overline{\Delta L}} \Rightarrow \Delta k = k \times \frac{\overline{\Delta L}_{\text{unc}}}{\overline{\Delta L}}$$

From Equation 2, we know $k = \frac{mg}{\Delta L}$. Re-writing we get:

$$\Delta k = \frac{mg\overline{\Delta L}_{\text{unc}}}{[\overline{\Delta L}]^2}$$

Hence, it is notable to state the uncertainty is k (Δk) rises based on the increase of $(\Delta L)^2$, evident as to say why the error bars in Figure 4 rise significantly as T rises in increment. The absolute uncertainty of T (ΔT) on the other hand, was determined to be $\pm 0.1°\text{C}$ and are not visible on the graph (Figure 4), thus deemed negligible.

Furthermore, to determine percentage error, the investigation would require literature values, derived from a theoretical equation. Since there is no direct equation that relates temperature T to spring constant k (of a rubber band in particular too), percentage error calculation is deemed unattainable.

Lastly to calculate the percentage uncertainty of the gradient, the line of best-fit, min and max lines are used:

$$m_{\text{min}} = 0.365\,\text{Nm}^{-1} \cdot^{\circ} \text{C}^{-1}, m_{\text{best}} = 0.426\,\text{Nm}^{-1} \cdot^{\circ} \text{C}^{-1}, m_{\text{max}} = 0.510\,\text{Nm}^{-1} \cdot^{\circ} \text{C}^{-1}$$

$$\Delta m = \frac{\left(\frac{m_{\text{max}} - m_{\text{min}}}{2}\right)}{m_{\text{best}}} \times 100 = \frac{\frac{0.510 - 0.365}{2}}{0.426} \times 100 = 17\%$$

Evaluation

With certain control variables taken into consideration, there were a few assumptions that hindered the validity of the procedure, and the investigation's methodology (in particular the scope, and limit of the experiment). It is critical to observe and take into account different sources of (systematic and random) error to gain a better understanding of the data collected.

Hooke's Law

One argument is that rubber bands don't quite follow Hooke's law beyond a certain extension. Hooke's law requires a linear relationship between F_a and Δx, according to Equation 1. The nature of the underlying polymeric chains show that with relatively small extensions (deformations), rubber demonstrates Hookean behavior, but turns into an S-like curve when F and Δx are plotted.

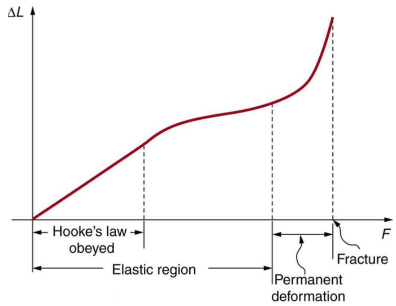


Figure 6: $F - \Delta L$ graph (lumencandela, n.d)

This detail hinders the certainty in which one measures the "elasticity" of a rubber band (a methodological issue), and whether spring constant (through Hooke's Law) is an accurate method in the investigation. One modification would be to use a smaller hook weight, in order to significantly decrease ΔL, in which the rubber band extends to.

Sources of Error

Sources of error and its effects	**Significance**	**Improvements**
During the experimental procedure, the temperature of the rubber band could not be controlled during the process of transferring the band to the retort stand from either the cooktop/fridge. This is the presence of random error, as the cooling/heating of certain items are proportional to the difference in temperature to the environment, demonstrated by Newton's Law of Cooling.	Maintaining the temperature is critical to the integrity of the experiment, as this would lead to anomalous data collected.	An alternative would be to carry out the thermal process without detaching the rubber band, with the use of a hairdryer, and likewise placing the apparatus in a system of cold temperature itself (inside a freezer/cold room) and assess change in length.

Retort stand: The DIY built stand holds its own weaknesses as the horizontal beam to which the ruler and rubber band is attached to, is prone to stress over time, and doesn't form a perfect right angle with the vertical beam. This would be a systematic error present in the experiment, as the structural weakness would apply to all data points.	Low significance: The change in length between a complete right angle would be minimal, thus affecting the results to a small extent.	Use a new stand, perhaps one metal, which is fairly attainable in a school-lab setting, and which can ensure a 90 degree angle (as it is machine manufactured). It is also important to fasten the ruler and hook to the new retort-stand.
Ruler: From figure 3, an evident gap between the rubber band and ruler is present due to the enlarged weight required to create a pulling force. Due to this, a certain amount of estimation was required to produce a length, hence a case random error.	High significance due to the large gap, readings may be accurate/inaccurate for certain increments, impacting the significance on data collected.	In addition to the hook attached to the rubber band, a low-mass horizontal beam (perpendicular) can be fixed to indicate the reading on the ruler, as shown in the illustration in Appendix II. However this requires careful consideration, as the mass of the horizontal beam should not affect the forces on the rubber band.

Extensions and Further Investigations

This investigation explored the relationship between temperature (°C) and spring constant (Nm^{-1}) of a rubber band. The overall methodology as well as the experimental procedure can be modified and extended in ways to develop new/better understandings of the topic at hand. Firstly, the aforementioned improvements for different sources of error can be addressed, so that more accurate results can be drawn. Methodological extensions include adapting the independent variable (IV). The IV can be extended to fit a wider range to better assess the correlation between the two variables. Factors that affect spring constant can be altered to other options such as qualitative variables such as color, or quantitative ones such as cross-sectional area of a rubber band.

To determine the breaking point of the rubber band, as it poses important physical meaning, an extension to the procedure can be added. As discussed in Physical Interpretation, this symbolizes the domain of the model. By gradually testing extreme temperatures, we can deem the limit of when the rubber band would become permanently deformed (i.e, the rubber band may snap when placed under the same force), as experimentation of any temperature further than the bounds would become pointless.

References

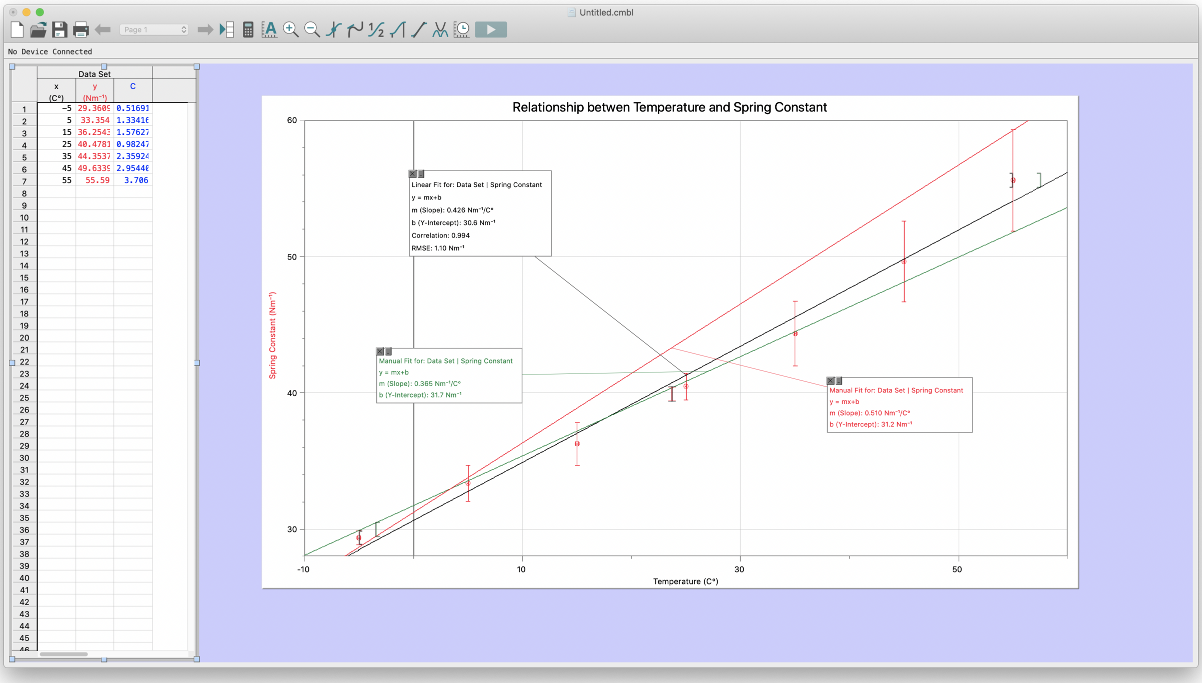
Jansen, K. M. B. (n.d.). (PDF) thermomechanical modelling and characterisation of ... Retrieved March 6, 2022, from https://www.researchgate.net/publication/331683452_Thermomechanical_modelling_and_characterisation_of_polymers_version

Anthony, R. L., Caston, R. H., & Guth, E. (n.d.). Equations of state for natural and synthetic rubber-like materials. I. Unaccelerated Natural Soft Rubber. ACS Publications. Retrieved March 5, 2022, from https://pubs.acs.org/doi/10.1021/j150422a005

Boundless. (n.d.). Boundless physics. Lumen. Retrieved March 5, 2022, from https://courses.lumenlearning.com/boundless-physics/chapter/elasti\hangindent=0.5cmcity-stress-strain-and-fracture/

Appendices

Appendix I: Logger Pro for Data Analysis



Appendix II: Horizontal Beam for Apparatus